THE CATHOLIC UNIVERSITY OF EASTERN AFRICA

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MAIN EXAMINATION

MAY – JULY 2019 TRIMESTER

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE

SPECIAL / SUPPLEMENTARY EXAMINATION

MAT 263: PROBABILITY AND STATISTICS IV

Date: JULY 2019	Duration: 2 Hours
INSTRUCTIONS: Answer Question ONE and any other TWO Questions	

QUESTION ONE (30 MARKS)

a) Suppose that three random variables X_1, X_2 and X_3 have a continuous density function given by

$$f(x_1, x_2, x_3) = \begin{cases} c(x_1 + 2x_2 + 3x_3), 0 < x_i < 1, i = 1,2,3 \\ 0, elsewhere \end{cases}$$

Determine:

i. The constant c

ii. The marginal p.d.f of X_1 and X_2

iii. $\Pr\left(x_3 < \frac{1}{2} / x_1 = \frac{1}{4}, x_2 = \frac{3}{4}\right)$ (8 marks)

b) A random vector $\underline{X} = (X_1, X_2, X_3, X_4)'$ has the 4-variate p.d.f given by $f(\underline{x}) = \begin{cases} 16x_1x_2x_3x_4, 0 < x_i < 1, i = 1,2,3,4 \\ 0, elsewhere \end{cases}$.

Calculate the probability that $\underline{X}$ lies in the region

$D = \{(x_1, x_2, x_3, x_4); x_1 < \frac{1}{2}, x_4 > \frac{1}{3}\}$. (5marks)

c) Let y be distributed $N_3(\mu, \Sigma)$, where $\mu = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$.
Find the following:

i. The distribution of $z = 4y_1 - 6y_2 + y_3$ (3 marks)

ii. The distribution of $z = \begin{pmatrix} y_1 - y_2 + y_3 \\ 2y_1 + y_2 - y_3 \end{pmatrix}$ (4 marks)

iii. $f(y_2 / y_1, y_3)$ (4 marks)

d) Let $\overline{X}$ and S^2 be the mean and variance of a random sample of 25 from a distribution which is normal with mean 3 and variance 100.

Compute $\Pr(0 < \overline{X} < 6, 49.6 < S^2 < 145.6)$ (6 marks)

QUESTION TWO (20MARKS)

a) Let $\underline{Y}$ be a random vector distributed as $N_p(\mu, \Sigma)$. Let $\underline{Y} = (\underline{Y}_1, \underline{Y}_2)$ when $\underline{Y}_1$ and $\underline{Y}_2$ are $r \times 1$ and $s \times 1 (r + s = p)$ random vectors:

i. Determine the distribution of $\underline{Y}_1$ and $\underline{Y}_2$ (6 marks)

ii. Show that $Y_i \sim N[\mu_i, \sigma_{ii}], i = 1, 2, \ldots p$. Where Y_i is the i^{th} component of $\underline{Y}$ $\mu_i = \mathrm{E}(Y_i)$ and $\Sigma = (\sigma_{ij})$.

(4 marks)

b) Let the random vector $\underline{X}$ have $N_2[\mu, \Sigma]$ distribution with its pdf $f(x) = c\exp\left\{-\frac{1}{2}Q(\underline{x})\right\}$ where c is some constant and $Q(\underline{x}) = x_1^2 + 2x_2^2 - x_1x_2 - 3x_1 - 2x_2 + 4$, find:

i. c (3 marks)

ii. μ (3 marks)

iii. Σ (4 marks)

QUESTION THREE (20MARKS)

a) Suppose that a three-dimensional random vector $\underline{X} = (x_1, x_2, x_3)$ has moment generating function

$$M(t_1, t_2, t_3) = \exp\left[\frac{1}{2}\left(5t_1^2 + 3t_2^2 + 2t_3^2 + 4t_1t_2 + 6t_1t_3\right)\right]$$

Determine:

i. The matrix $\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^3 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$ of the variances and covariance's $\underline{X}$ and hence obtain the matrix $\begin{pmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{pmatrix}$ of the correlations between the components of the random vector $\underline{X}$ (6 marks)

ii. The marginal distribution of X_1 and X_2 and write down its density function (4 marks)

iii. The correlation coefficient between $Y = X_1 + X_2$ and $Z = X_1 + X_2 + X_3$ (4 marks)

b) Suppose that the random vector $\underline{X} = (X_1, X_2, \ldots X_p)'$ has a continuous distribution whose p.d.f is $f(x_1, x_2, \ldots x_p) = \left(\frac{1}{2\pi}\right)^{p/2} e^{-\frac{1}{2}\sum_{i=1}^{n} x_i}; -\infty < x_i < \infty$.

Find the moment generating function of $\underline{X}$. (6 marks)

QUESTION FOUR (20MARKS)

a) Suppose that the lifetimes of light bulbs produced in a certain factory are distributed according to the pdf $f(x) = xe^{-x}$, $x > 0$. Let $X_1, X_2, \ldots, X_n$ denote the lifetimes of a random sample of n light bulbs drawn from a factory production. Determine:

i. The joint p.d.f of $X_1, \ldots, X_n$

ii. $\Pr(X_1 < X_2 < \ldots < X_n)$ (8 marks)

b) State and prove the central limit theorem (12 marks)

QUESTION FIVE (20MARKS)

a) Suppose that the random vector has the 3-variate distribution given by

$$f(x_1, x_2, x_3) = \frac{20!}{x_1!x_2!x_3!(20 - x_1 - x_2 - x_3)!}\left(\frac{1}{10}\right)^{x_1}\left(\frac{1}{5}\right)^{x_2}\left(\frac{3}{10}\right)^{x_3}\left(\frac{2}{5}\right)^{20-x_1-x_2-x_3}$$

i. Obtain the moment generating function $\underline{X}$ (3 marks)

ii. Determine the marginal distribution of X_2 and X_3 (4 marks)

iii. The conditional mean and variance of X_1 given that $X_2 = X_2$ and $X_3 = X_3$ (5 marks)

b) Suppose that $X_1, X_2, \ldots X_n$ is a random sample from a population with mean μ and variance σ^2. Let $Q = \sum_{i=1}^{n}(X_i - \overline{X})^2$ where $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. Show that $E(Q) = (n-1)\sigma^2$

(8 marks)

END