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MAIN EXAMINATION

SEPTEMBER – DECEMBER 2020 SEMESTER

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS AND ACTUARIAL SCIENCE

REGULAR PROGRAMME

MAT 543: PROBABILITY THEORY

Date: DEC 2020	**Duration: 3 Hours**
INSTRUCTIONS: Answer any THREE Questions	

QUESTION ONE

a) Let $(\Omega, \mathcal{F}, P)$ a measure space? Define each term and use illustrations where possible (6 marks)

b) Let $\Omega = \mathbb{R}, \mathcal{F} = \mathcal{B}(\mathbb{R})$ and

$$A_n = \begin{cases} \left[0, \frac{1}{n}\right], \text{if } n \text{ is odd} \\ \left[1 - \frac{1}{n}, 1\right], \text{if } n \text{ is even} \end{cases}$$

Show that $\limsup A_n = \{0,1\}$ and $\liminf A_n = \varnothing$ (5 marks)

c) Let be a non-empty set and $\mathcal{F} = \mathcal{P}(\Omega)$ the power set of Ω. Define a measure μ on $(\Omega, \mathcal{P}(\Omega))$ by

$$\mu(\varnothing) = 0$$
$$\mu(E) = +\infty \text{ if } E \neq \varnothing$$

i) Show that μ is a measure (3 marks)

ii) Show that μ is neither finite nor $\sigma-$finite (3 marks)

d) Define an expectation of a random variable, hence or otherwise show that the expectation operator is linear (4 marks)

e) State the Central Limit Theorem (2 marks)

QUESTION TWO

a) State and prove the Dominated Convergence theorem (7 marks)

b) If f and g are measurable functions, show that $\left(\frac{f}{g}\right)^2$ is also measurable (5 marks)

c) Define the following terms:
   i) Convergence almost everywhere
   ii) Convergence in probability
   iii) Convergence in distribution (6 marks)

d) Let $(\Omega, \mathcal{F})$ be a measurable space and $A \subset \Omega$ such that $A \notin \mathcal{F}$ .Show that the indicator function $1_A(\omega)$ is not a measurable function (5 marks)

QUESTION THREE

a) Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of $\sigma-$algebras on a non-empty set Ω. Show that their countable intersection is also a $\sigma-$algebra (6 marks)

b) Define a measurable function. Further, show that given two measurable functions f and g, their difference $f-g$ is also measurable (6 marks)

c) Let $\{B_n\}_{n\geq 1}$ be a sequence of events. Prove that if $B_n \uparrow B$, then $P(B_n) \uparrow P(B)$ as $n \to \infty$ (7 marks)

d) Using simple random variables, show that the expectation operator is monotone i.e. if $X \leq Y$, then $E(X) \leq E(Y)$ (4 marks)

QUESTION FOUR

a) Prove that if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ does not always imply that $X_n + Y_n \xrightarrow{d} X + Y$ (7 marks)

b) State and prove the weak law of large numbers (6 marks)

c) State and prove the first Borel Cantelli's lemma (7 marks)

d) Let Ω be a non-empty. Using a counter example, show that a union of $\sigma-$algebras on the set Ω is not necessarily a $\sigma-$algebra (3 marks)

QUESTION FIVE

a) Let $X \sim B(n, p)$ a Binomial random variable with the given fixed parameters. Given a fixed $a > 0$, show that

$$P\left(\left|\frac{X}{n} - p\right| \geq a\right) \leq \frac{p(1-p)}{a^2 n}$$ (6 marks)

b) Prove that the Dirichlet function on the interval $[0,1]$ is Lebesgue integrable but not Riemann integrable (4 marks)

c) Prove that if $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$, X_n and Y_n are independent, then $X_n Y_n \xrightarrow{d} XY$ (6 marks)

d) Let $(\Omega,\mathcal{F})$ be a measurable space. Show that given two measurable f and g functions on $(\Omega,\mathcal{F})$, their product $f \times g$ is also measurable (5 marks)

e) Show that if P is a probability measure on the measurable space $(\Omega,\mathcal{F})$, then P is additive (3 marks)

END