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MAIN EXAMINATION

SEPTEMBER –DECEMBER 2021

FACULTY OF SCIENCE

DEPARTMENT OF MATHEMATICS

REGULAR PROGRAMME

MAT 467: MULTIVARIATE ANALYSIS

Date: DECEMBER 2021	Duration: 2 Hours
INSTRUCTIONS: Answer Question ONE and any TWO Questions	

Q1.

a) Suppose that three random variables X_1, X_2 and X_3 have a continuous density function given by

$$f(x_1, x_2, x_3) = \begin{cases} c(x_1 + 2x_2 + 3x_3), 0 < x_i < 1, i = 1, 2, 3 \\ 0, elsewhere \end{cases}$$

Determine:

i. The constant c

ii. The marginal p.d.f of X_1 and X_2

iii. $\Pr\left(x_3 < \frac{1}{2} / x_1 = \frac{1}{4}, x_2 = \frac{3}{4}\right)$ **(6 marks)**

b) Show that $\Sigma = Var(\underline{X}) = \mathrm{E}(\underline{X}\underline{X}^T) - \underline{\mu}\underline{\mu}^T$ **(4marks)**

c) Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a population with mean μ and variance σ^2. Let $Q = \sum_{i=1}^{n} \left(X_i - \overline{X} \right)^2$ where $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. Show that $E(Q) = (n-1)\sigma^2$. **(5marks)**

d) Suppose that $\underline{X} \sim N_p\left(\underline{\mu}, \Sigma\right)$, $\Sigma > 0$. Let $Q = \left(\underline{X} - \underline{\mu}\right)^T \Sigma^{-1} \left(\underline{X} - \underline{\mu}\right)$. Show that $Q \sim \chi^2(p)$

(5marks)

e) Let $\underline{X}$ be such that $\mathrm{E}(\underline{X}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $Var(\underline{X}) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$

Find $\mathrm{E}(X_1^2 + 2X_2^2 + 3X_3^2 + 3X_1X_2 + 4X_1X_3)$ **(5marks)**

f) Suppose $\underline{X} = \left(X_1, X_2, X_3 \right)^T$ is a random vector with mean $\underline{\mu} = \left(\mu_1, \mu_2, \mu_3 \right)^T$ and $Var(\underline{X}) = \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & 0 \\ \rho^2 & 0 & 1 \end{pmatrix}$. Find $Var.\left[\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} / X_2 \right]$ **(5marks)**

Q2.

a) The covariance matrix corresponding to scaled (standardized) variables x_1, x_2 is

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Compute the Principal components explaining respectively $\frac{100(1+\rho)}{2}\%$ and $\frac{100(1-\rho)}{2}\%$ of the total variation. **(7marks)**

b) Let X be a $p-\text{variate}$ random vector with mean μ and variance-covariance matrix $\Sigma=((\sigma_{ij}))$;$i,j=1,2,\ldots,p$. Show that $E(X'AX)=tr(A\Sigma)+\mu' A\mu$ where A is a $p\times p$ symmetric matrix of constants **(7marks)**

c) Let X_1 and X_2 be jointly distributed random variables with means $\mu_1=\mu_2=0$ and variances $\sigma_1^2=2$, $\sigma_2^2=3$ and correlation $\rho=0.6$. Find the correlation between the linear functions $Y=4X_1+5X_2$ and $Z=3X_1+2X_2$.

(6marks)

Q3.

a) Let $\underline{X}$ denote a 3-variate random vector with $Var(\underline{X})=\begin{pmatrix}3&1&0\\1&2&0\\0&0&1\end{pmatrix}$. Let $\underline{V}=\begin{pmatrix}v_2\\v_1\end{pmatrix}$ where $v_1=X_1+X_2+2X_3$ and $v_2=X_1+4X_3$.Determine

(i) $Var(\underline{V})$ **(4marks)**

(ii) Corr.(v_1,v_2) **(2marks)**

b) Let $X_1,X_2,\ldots,X_n$ be $i.i.d$ random variables with mean μ and variance σ^2.

Let $Q(\underline{X})=(X_1-X_2)^2+(X_2-X_3)^2+\ldots+(X_{n-1}-X_n)^2$.

Show that $E(Q)=2(n-1)\sigma^2$ and $\frac{Q}{2(n-1)}$ is an unbiased estimator of σ^2 **(6marks)**

c) Let $\underline{Y}\sim N_p(\underline{\mu},\Sigma)$. Let $\underline{Y}^T=(\underline{Y}_1,\underline{Y}_2)$ where $\underline{Y}_1$ and $\underline{Y}_2$ are $r\times 1$ and $s\times 1$, $(r+s=p)$ random vectors.

i. Determine the distribution of $\underline{Y}_1$ and $\underline{Y}_2$ **(4marks)**

ii. Show that $Y_i\sim N[\mu_i,\sigma_{ii}]$;$i=1,2,\ldots,p$, where Y_i is the i^{th} component of $\underline{Y}$, $\mu_i=E(Y_i)$ and $\Sigma=(\sigma_{ij})$. **(4marks)**

Q4.

a) Let $\underline{X}$ be a 3×1 random vector with $Variance-\text{cov}\,ariance$ matrix, $\Sigma=\begin{bmatrix}5&2&3\\2&3&0\\3&0&2\end{bmatrix}$. Find

i. The variance of $Z=X_1-2X_2+X_3$ **(5marks)**

ii. The $Variance-\text{cov}\,ariance$ matrix of the random vector $\underline{Y}=(Y_1,Y_2)^T$ where $Y_1=X_1+X_2$ and $Y_2=X_1+X_2+X_3$ **(6marks)**

b) Let $X_i; i=1,2,...,n$ be iid random vectors from $N_p[\mu,\Sigma]$. Given that $\underline{\bar{X}}$ and S are M.L.E's of μ and Σ respectively.

(i) Write down the typical elements of the vector $\underline{\bar{X}}$ and matrix S. **(2marks)**

(ii) Give the expression for the Hotelling T^2 statistic and explain how you use it to test the hypotheses $H_0:\mu=\mu_0$ v/s $H_1:\mu\neq\mu_0$ **(7marks)**

Q5.

a) Suppose that y is $N_4[\mu,\Sigma]$ where

$$\mu=\begin{pmatrix}1\\2\\3\\-2\end{pmatrix},\quad \Sigma=\begin{pmatrix}4&2&-1&2\\2&6&3&-2\\-1&3&5&-4\\2&-2&-4&4\end{pmatrix}$$

Find

(i) the joint marginal distribution of y_1 and y_3 **(3marks)**

(ii) $\rho_{13\cdot24}$ **(5marks)**

b) Suppose that $\underline{X} = (X_1, X_2, X_3)$ has the 3-variate normal distribution with density

$$f(\underline{x}) = c\exp.\left(-\frac{1}{2}Q\right)$$ where

$$Q = \frac{1}{17}\left(11x_1^{\ 2} + 7x_2^{\ 2} + 5x_3^{\ 2} - 6x_1x_2 - 4x_1x_3 - 2x_2x_3 + 2x_1 - 16x_2 - 22x_3 + 18\right).$$

i. Identify $\underline{\mu}$ and $\underline{\Sigma}$. Hence find c. **(7marks)**

ii. Determine the distribution of $\underline{Q}$ and hence find k such that $\Pr.(Q > k) = 0.05$ **(5marks)**

END